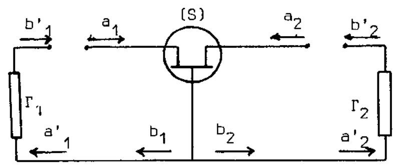

Fig. 3. Two-port FET oscillator circuit.

Verification for a Two-Port Loaded by Two Impedances

In Fig. 3, for the active device:

$$[S]=\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{21} \end{bmatrix} \tag{14}$$

and for the embedding circuit:

$$[S']=\begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix}. \tag{15}$$

The oscillation condition from (7) is

$$|M|=|[S][S']-[1]|$$

$$=\begin{vmatrix} S_{11}\Gamma_1-1 & S_{12}\Gamma_2 \\ S_{21}\Gamma_1 & S_{22}\Gamma_2-1 \end{vmatrix}=0 \tag{16}$$

which gives

$$(S_{11}\Gamma_1-1)(S_{22}\Gamma_2-1)-S_{12}S_{21}\Gamma_1\Gamma_2=0. \tag{17}$$

This equation results in the following two well-known conditions [1]:

$$S_{11}+\frac{S_{12}S_{21}\Gamma_2}{1-S_{22}\Gamma_2}=\frac{1}{\Gamma_1} \tag{18}$$

$$S_{22}+\frac{S_{12}S_{21}\Gamma_1}{1-S_{11}\Gamma_1}=\frac{1}{\Gamma_2}. \tag{19}$$

Three-Port Loaded by Three Impedances

In Fig. 4 for $[M]=[S][S']-[1]$ to be singular, we have

$$\begin{vmatrix} S_{11}\Gamma_1-1 & S_{12}\Gamma_2 & S_{13}\Gamma_3 \\ S_{21}\Gamma_1 & S_{22}\Gamma_2-1 & S_{23}\Gamma_3 \\ S_{31}\Gamma_1 & S_{32}\Gamma_2 & S_{33}\Gamma_3-1 \end{vmatrix}=0 \tag{20}$$

or

$$\frac{S_{12}S_{21}\Gamma_1\Gamma_2}{(1-S_{11}\Gamma_1)(1-S_{22}\Gamma_2)}+\frac{S_{13}S_{31}\Gamma_1\Gamma_3}{(1-S_{11}\Gamma_1)(1-S_{33}\Gamma_3)}+\frac{S_{23}S_{32}\Gamma_2\Gamma_3}{(1-S_{22}\Gamma_2)(1-S_{33}\Gamma_3)}+\frac{\Gamma_1\Gamma_2\Gamma_3(S_{12}S_{23}S_{31}+S_{21}S_{32}S_{13})}{(1-S_{11}\Gamma_1)(1-S_{22}\Gamma_2)(1-S_{33}\Gamma_3)}=1. \tag{21}$$

This is the same relation as is obtained by calculating in the classical way at each of the three-ports the following relation:

$$S''_{11}\Gamma_1=S''_{22}\Gamma_2=S''_{33}\Gamma_3=1$$

where S''_{11} is the modified reflection coefficient at port 1 with ports 2 and 3 loaded by impedances corresponding to refl. coeff. Γ_2 and Γ_3.

It may be noted that though in both the above examples the transfer scattering parameters of the embedding network have been taken as zero, the approach presented is equally applicable to analyse complex embedding networks with nonzero transfer scattering parameters, for example a YIG sphere coupled to both gate and source of an FET [4].

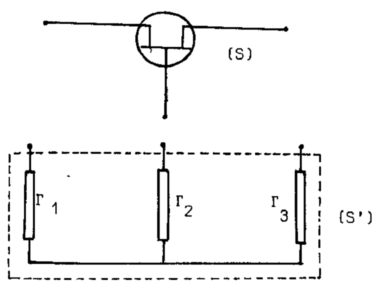

Fig. 4. Three-port FET oscillator circuit.

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A Simple Numerical Method for the Cutoff Frequency of a Single-Mode Fiber with an Arbitrary Index-Profile

ANURAG SHARMA AND A. K. GHATAK

***Abstract*—A simple numerical method for calculating the cutoff frequency of single-mode operation in optical fibers with an arbitrary index-profile is presented. The method does not involve any approximation other than the scalar approximation and is applicable even to numerical data from index-profile measurements. The calculations are simple and can be carried out even on a programmable calculator.**

I. Introduction

The cutoff frequency of single-mode operation in optical fibers is an important parameter since it defines the upper limit on the diameter of a single-mode fiber. However, the cutoff condition cannot be obtained analytically except in the case of step-index [1], parabolic-index [2], and W type fibers [3], [4] and, as such, various approximate [5]–[7] and numerical [8]–[13] methods have been developed to calculate cutoff frequencies of various other types of graded-index fibers. Of the approximate methods, the variational method [5] gives only an accuracy of the order of 1 percent in the calculation of the cutoff frequency. The perturbation method [6] gives good results only for profiles which are nearly parabolic and involves the evaluation of higher transcendental functions such as confluent hypergeometric function [15]

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besides the numerical evaluation of integrals. The integral equation method [7] involves successive approximations which require an initial approximation of the field and numerical evaluation of double integrals at each stage.

On the other hand, the direct numerical methods either are highly involved and time consuming because of intensive computations or are applicable to a limited class of index-profiles. For example, the power-series method [8], [9] is applicable only to the profiles which can be expressed as a finite power series. Other numerical methods [10]–[13] involving the "staircase approximation," [10], [11], the finite element technique [13], and power series expansion [12] require extensive computation and large data storage and thus, are suitable only for large computers.

We have presented here a simple numerical method which can be applied to any arbitrary-index profile and even to numerical data from the index-profile measurements and does not involve any approximation other than the scalar approximation. The computation required can be carried out even on a programmable calculator.

II. Method

The index profile of a graded-index fiber can, in general, be written as

$$n^2(R)=n_1^2-\Delta(n_1^2-n_2^2)f(R), \qquad R\leqslant 1$$
$$=n_2^2, \qquad R>1 \tag{1}$$

where $R=r/a$, a being the radius of the core, n_1 is the maximum refractive-index in the core, n_2 the refractive-index of the cladding, Δ determines the refractive-index jump at the core-cladding interface and $0\leqslant f(R)\leqslant 1$ defines the shape of the profile in the core. The modal field, $\Psi(R)$, of the first higher mode (TE_{01} mode), in the core, is given by the following scalar wave equation:

$$\frac{d^2\Psi}{dR^2}+\frac{1}{R}\frac{d\Psi}{dR}+\left[U^2-V^2\Delta f(R)-\frac{1}{R^2}\right]\Psi(R)=0, \qquad R\leqslant 1 \tag{2}$$

and the field, in the cladding, is given by

$$\Psi(R)\sim K_1(WR), \qquad R\geqslant 1 \tag{3}$$

where K_n is the modified Hankel function [15] and

$$V^2=k^2a^2(n_1^2-n_2^2)$$
$$U^2=(k^2n_1^2-\beta^2)a^2 \tag{4}$$
$$W^2=V^2-U^2$$

β is the propagation constant and k is the free-space wavenumber. The field, $\Psi(R)$, satisfies the following boundary conditions (see Appendix):

$$\Psi(0)=0 \tag{5}$$

$$\frac{1}{\Psi}\frac{d\Psi}{dR}\bigg|_{R=1}=-\frac{WK_0(W)}{K_1(W)}-1. \tag{6}$$

At cutoff $\beta\to kn_2$ so that $W\to 0$ and $U\to V$ and (2) for the field $\Psi(R)$ becomes

$$\frac{d^2\Psi}{dR^2}+\frac{1}{R}\frac{d\Psi}{dR}+V_c^2[1-\Delta f(R)]\Psi-\frac{\Psi}{R^2}=0, \qquad R\leqslant 1 \tag{7}$$

where V_c is the normalized cutoff frequency. The boundary condition (6) now takes the form

$$\frac{1}{\Psi}\frac{d\Psi}{dR}\bigg|_{R=1}=-1. \tag{8}$$

The second-order differential equation (7) can be reduced to a first-order differential equation by making the following substitution:

$$G(R)=\frac{1}{\Psi}\frac{d\Psi}{dR}. \tag{9}$$

Thus (7) can be written as

$$\frac{dG}{dR}=-G^2-\frac{G}{R}+\frac{1}{R^2}-V_c^2[1-\Delta f(R)] \tag{10}$$

and boundary conditions (5) and (8) as

$$G(0)=\infty \text{ and } G(1)=-1. \tag{11}$$

In order to avoid the boundary condition $G(0)=\infty$, we define a new function $F=1/G$ so that

$$F(0)=0. \tag{12}$$

The function $F(R)$ satisfies the following equation:

$$\frac{dF}{dR}=1+\frac{F}{R}-\frac{F^2}{R^2}+F^2V_c^2[1-\Delta f(R)]. \tag{13}$$

Further, it may be noted that at $R=0$ the right-hand side (RHS) of (13) is indeterminate and one has to take its limiting value. It can be easily shown that (see Appendix)

$$\lim_{R\to 0}\frac{dF}{dR}=1. \tag{14}$$

Thus the problem of finding the cutoff frequency has reduced to solving the transcendental equation

$$S(V_c)=-1 \tag{15}$$

where

$$S(V_c)=G(R)|_{R=1} \tag{15a}$$

and $G(R)|_{R=1}$ for a particular value of V_c is obtained by solving (10) and (13) using an appropriate numerical method, e.g., the fourth order Runge–Kutta method [16]. To begin, in the region of $R\simeq 0$, (13) is used with appropriate conditions, viz., (12) and (14), because, at $R=0$, $G\to\infty$. As soon as F exceeds unity one makes a switch over to (10)[1] with the condition that $G=1/F$ at the corresponding value of R and calculates the $G(R)$ at $R=1$ which gives the LHS of (15).

III. Numerical Examples and Discussion

We present here some numerical examples in order to show the effectiveness of our method. All the calculations have been carried out on a microcomputer (ECIL Micro 78, based on INTEL 8080 microprocessor) using PL/S programming language.[2]

We first consider a parabolic-index fiber. The cutoff condition, in this case, can be obtained analytically and is given by [2]

$$\frac{V_c}{2}M\left(1-\frac{V_c}{4},2,V_c\right)=\left(2-\frac{V_c}{2}\right)M\left(2-\frac{V_c}{2},2,V_c\right) \tag{16}$$

where $M(a,b,z)$ is the confluent hypergeometric function [15]. We have compared the computation time required to solve (16) with the time required to solve (15) using the same transcendental-equation solving routine. Using (16) it takes about 65 s to obtain the cutoff frequency with an accuracy of 0.001 percent

[1]The switch over is necessary because at some value of R in the range 0 to 1, $F\to\infty$ and $G\to 0$ (which corresponds to the maximum value of the modal field in the core). The switch over is made around $F\simeq 1$ in order to minimize the truncation error in the computation of $G=1/F$.

[2]PL/S, implemented by SOFTEK Private, Ltd., is a modified version of PL/1.

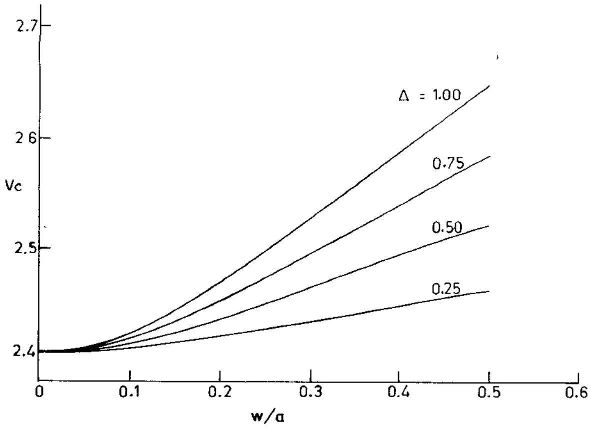

Fig. 1. The cutoff frequency V_c versus the normalized width of the Lorentzian dip w/a for different values of Δ for a step-index fiber.

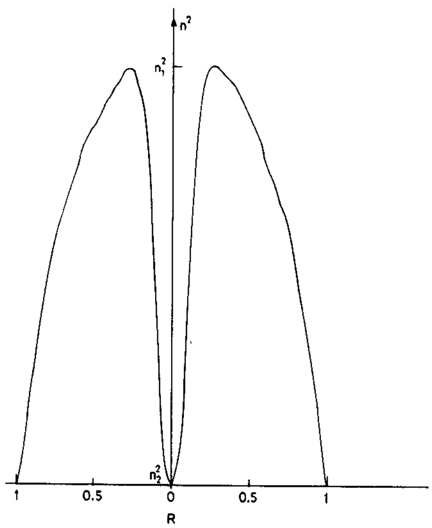

Fig. 2. A typical experimental refractive-index profile (redrawn from fig. 4 of [14]).

while using (15) it takes about 94 s. Thus although our method is a numerical method, the time taken is of the same order as that required to solve the analytical transcendental equation. Further, analytical expressions for cutoff condition are possible only for a few index profiles such as step-index $[f(R)=0]$ and parabolic-index $[f(R)=R^2]$, whereas our method is applicable to any arbitrary index-profile. It may be mentioned here that the power series method [8], [9] is faster than our method but is applicable only to those profiles which can be expressed as a finite power series, e.g., it is applicable only to those α-profiles for which α is an integer.

Next, we present the results of two calculations to show the applicability of our method to any arbitrary–analytical or numerical–profile. First, we consider an analytical profile—a step-index fiber with a Lorentzian dip, for which

$$f(R)=\frac{(w/a)^2(1-R^2)}{(w/a)^2+R^2} \tag{17}$$

where w is the width of the dip. In Fig. 1, we have plotted the cutoff frequency V_c, as a function of the normalized width, w/a, for different values of Δ. Our results for $\Delta=1$ agree very well with those obtained by using the more cumbersome integral-equation method [7].

Finally, we consider a typical experimentally measured profile shown in Fig. 2. The R-axis was divided into 70 equal parts and the corresponding profile values were taken as the profile data. The cutoff frequency was calculated to be $V_c=2.2036$.

Although, for the above calculations we have used a microcomputer, these calculations can be carried out even on a programmable calculator (with about 1000 programming steps).

IV. Conclusion

In this paper, we have described a simple and exact numerical method to calculate the cutoff frequency of the first higher mode which determines the single-mode limit in optical fibers. The method is applicable to arbitrary index profiles and even to numerical data from index profile measurements. The method does not involve the computation of any special or elementary transcendental function and is suitable even for a programmable calculator of about 1000 programming steps.

Appendix
Boundary Conditions at $R=0$

The boundary conditions on Ψ and its derivatives at $R=0$ can be easily derived by expanding Ψ about $R=0$ as follows:

$$\Psi(R)=\sum_{n=0}^{\infty} a_n R^n \tag{A.1}$$

where

$$a_n=\frac{1}{n!}\left.\frac{d^n\Psi}{dR^n}\right|_{R=0} \tag{A.2}$$

The wave equation can be written as

$$\frac{1}{R}\frac{d}{dR}\left(R\frac{d\Psi}{dR}\right)-\frac{m^2\Psi}{R^2}+P(R)\Psi(R)=0 \tag{A.3}$$

where $m=0,1,2,\cdots$ is the azimuthal symmetry number and $P(R)=U^2-V^2\Delta f(R)$. Substituting (A.1) in (A.3), we get

$$-\frac{m^2a_0}{R^2}+\frac{1-m^2}{R}a_1+\sum_{n=0}^{\infty}\Big[\{(n+2)^2-m^2\}a_{n+2}+P(R)a_n\Big]R^n=0. \tag{A.4}$$

In order that the field Ψ be finite at $R=0$, the following conditions must be satisfied:

$$a_0=0, \quad \text{for } m\neq 0 \tag{A.5}$$

$$a_1=0, \quad \text{for } m\neq 1 \tag{A.6}$$

and

$$\left[(n+2)^2-m^2\right]a_{n+2}+P(R)a_n=0. \tag{A.7}$$

The above conditions would require

$$\frac{d^p\Psi}{dR^p}\neq 0, \quad \text{for } p=m, m+2, m+4,\cdots$$
$$=0, \quad \text{otherwise.} \tag{A.8}$$

Thus for $m=1$, we have

$$\Psi(0)=0 \tag{A.9}$$

which justifies (5). Further,

$$\left.\frac{d\Psi}{dR}\right|_{R=0}\neq 0 \text{ and } \left.\frac{d^2\Psi}{dR^2}\right|_{R=0}=0, \quad \text{for } m=1 \tag{A.10}$$

which on substitution in the following equation:

$$\frac{df}{dR}=1-\Psi\frac{d^2\Psi}{dR^2}\bigg/\left(\frac{d\Psi}{dR}\right)^2 \tag{A.11}$$

give (14).

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Microstrip Dispersion in a Wide-Frequency Range

EIKICHI YAMASHITA, KAZUHIKO ATSUKI, AND TETSUYA HIRAHATA

***Abstract*—The dispersion property of microstrip lines was measured in a frequency range from 2 to 50 GHz and compared with that estimated by an approximate formula in a previous paper.**

I. Introduction

The frequency dependence of wave velocity in a transmission line is called dispersion property, and is an important quantity when the line is used in a wide frequency range. The ratio of the propagation constant of the transmission line to that of free space β/β_0 is also used as a quantity to express the dispersion property. The dispersion property can be exactly analyzed by solving Maxwell's equations as a boundary value problem. The integral equation method [1] and the mode-matching method [2], among others, showed good agreement in the results of the analysis of microstrip dispersion. However, an approximate formula of the microstrip dispersion is needed in desk calculations and the CAD of microwave integrated circuits. Though a few approximate formulas have been reported in the past based on some physical considerations and experimental data for frequencies up to 12 GHz, [3]–[6], these empirical formulas have had narrow ranges of applicability. We also derived an approximate formula of microstrip dispersion [7] from the numerical result of the integral equation method [1].

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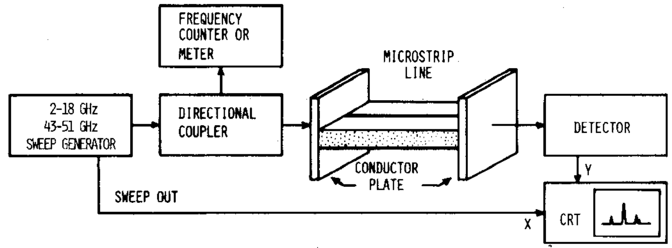

Fig. 1. Experimental setup for measuring the dispersion property of microstrip lines.

This paper describes the experimentally measured dispersion property of some microstrip lines in a wide-frequency range compared with that estimated by the above approximate formula.

II. Approximate Dispersion Formula

The approximate formula of microstrip dispersion given in a previous paper [7] is

$$\frac{\beta}{\beta_0}=\frac{\sqrt{\epsilon^*}-\dfrac{\beta_{TEM}}{\beta_0}}{1+4F^{-1.5}}+\frac{\beta_{TEM}}{\beta_0} \tag{1}$$

where

$$F=\frac{4h\sqrt{\epsilon^*-1}}{\lambda_0}\left[0.5+\left\{1+2\log_{10}\left(1+\frac{w}{h}\right)\right\}^2\right] \tag{2}$$

β_{TEM}	the propagation constant derived with the quasi-TEM wave approximation,
λ_0	wavelength in vacuum,
h	the height of the substrate,
w	the width of the strip conductor,
ϵ^*	the dielectric constant of the substrate.

The applicable ranges of this formula are

$$2<\epsilon^*<16$$
$$0.06<w/h<16$$
$$0.1 \text{ GHz}<f<100 \text{ GHz}.$$

Though the lowest usable frequency is limited by 0.1 GHz, the propagation constant for frequencies less than 0.1 GHz has been already given as β_{TEM}.

III. Experimental Results

The microstrip dispersion was measured with a resonance method. Fig. 1 shows an experimental setup for measuring the guide wavelength λ and the free space wavelength λ_0. The

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